CS 383
HW 4 Solutions

1. Remember quotient languages from HW 3: If L is a regular language over Σ and a $\in \Sigma$ then L/a is the set of strings w such that wa is in L. Either prove or disprove the following identities:
   a. (L/a)a = L No. Use L={011, 000} (L/1)1 = {011}
   b. (La)/a = L Yes. La={wa|w∈L}. (La)/a={α|αa∈La}={α|α=w for some w in L} = L

2. Suppose L is a regular language. Show that min(L) is also regular, where min(L)= {w | w is in L but no proper prefix of w is in L}

   Consider a DFA P for L. Let P' be P with all of the transitions out of final states removed. If string w is accepted by P' then it takes P to a final state, so it is accepted by P and must be in min(L) because it doesn't pass through (enter, then leave) any final states. On the other hand, if w is in min(L) it must take P to a final state without passing through any final states, so none of the transitions it needs are removed in P', which means that w is accepted by P' as well. In other words w is in min(L) if and only if w is accepted by P', so min(L) is regular.

3. Suppose L is regular. Show that prefix(L) is also regular, where prefix(L) = {w | wx is in L for some x (including x=ε)}. prefix(L) is the set of all prefixes of all strings in L. These don't need to be proper prefixes, so L is a subset of prefix(L)

   Start with a DFA for L with a minimum number of states. If there was a state in this DFA from which it was impossible to get to a final state we could remove it and get an even smaller DFA that accepted the same language, so every state in the minimal DFA can reach a final state. Now make a new DFA identical to the minimal one only have all of the states be final. This accepts string w if and only if w is a prefix of some string in L.

4. For any language L let powers(L) = {x^n | n >= 0 and x ∈ L}. Find an example where L is regular but powers(L) is not regular.

   Let L = 0*1. powers(L)={ $(0^j1)^n$| j>=0 and n>=0}. Suppose this is regular and let p be its pumping constant. Let w = 0^p10^p1. If we had a decomposition w=xyz satisfying the Pumping Lemma, the y portion would consist of just 0's so xy^2z would be $0^{p+j}10^p1$. This does not have the form w^n for any w in L. So our string w is not pumpable and powers(L) is not regular.

5. Design a context-free grammar for {0^n1^n | n>=1}
   S => 0S1 | 01

6. Design a context-free grammar for $\{a^i b^j c^k \mid i \mathrel{!=} j\}$

$S \Rightarrow AC$
$A \Rightarrow A_1 \mid A_2$
$A_1 \Rightarrow a\, A_1\, b \mid a\, A_3 \mid a$
$A_3 \Rightarrow a\, A_3 \mid a$
$A_2 \Rightarrow a\, A_2\, b \mid B$
$B \Rightarrow Bb \mid b$
$C \Rightarrow cC \mid c \mid \varepsilon$

There may be shorter solutions but this one is straightforward. A generates strings with either more a's than b's or fewer a's than b's. A_1 generates the former, A_2 the latter.

7. Here is a context-free grammar:
   $S \Rightarrow aS \mid Sb \mid a \mid b$
   Prove by induction on the string length that no string in the language represented by this grammar has ba as a substring.

I claim that every string of length n in this language does not have ba as a substring. It is certainly true of strings of length 1. Suppose this is true for all strings of length up to n, and w is a string in the language of length n+1. Then w must be $a\alpha$ where $S \overset{*}{\Rightarrow} \alpha$ and α has length n, or w must be βb where $S \overset{*}{\Rightarrow} \beta$ and β has length n. By the induction hypothesis neither α nor β contains ba as a substring. Preceding α by a can't introduce ba as a substring; following β with b can't introduce ba. So w also can't contain ba as a substring. So if strings of length up to n have no substring ba neither do strings of length n+1. By induction no strings in the language have ba as a substring.